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On complex singularities of solutions of the equation $\mathcal{H}u_x - u + u^p = 0$

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Abstract. It has been proved recently by Bona and Li Yi (Bona J L and Li Yi A 1997 *J. Math. Pure Appl.* **76** 377) that solitary wave solutions of a certain class of nonlinear nonlocal equations can be extended into the complex domain. In this paper we present an approach for the study of singularities of such extensions. This approach is applied to the localized solutions of the equation $\mathcal{H}u_x - u + u^p = 0$, $p = 3, 4, 5$ where $\mathcal{H}$ is the Hilbert transform. The location of the closest to real axis singularity $z = z_0$ was found numerically; the analysis of this type of singularity shows that in the vicinity of $z = z_0$ these complex extensions of solutions cannot be represented by the series $\frac{1}{(z-z_0)^\rho} \sum_{n=0}^{\infty} A_n(z - z_0)^n$, i.e. they do not correspond to some power of a meromorphic function.

1. Introduction

This paper is concerned with the nonlinear nonlocal equation

$$\mathcal{H}u_x - u + u^p = 0 \qquad p > 1 \qquad p \in \mathbb{Z} \tag{1}$$

where $\mathcal{H}$ is the Hilbert transform

$$\mathcal{H}u(x) \equiv \frac{1}{\pi} \,\mathrm{vp} \int_{-\infty}^{+\infty} \frac{u(x')}{x' - x} \,\mathrm{d}x'.$$

Equation (1) arises in various physical applications including lattice models with long-range interactions [1] and the theory of ferromagnets [2]. In particular, it describes travelling wave solutions $u(x) \equiv u(\xi - ct)$ of the generalized Benjamen–Ono equation

$$u_t + pu^{p-1}u_\xi + \mathcal{H}u_{\xi\xi} = 0. \tag{2}$$

In the case $p = 2$, equation (2) is integrable, and its soliton solutions can be described in terms of the dynamics of poles of $u(\xi, t)$ in the complex plane [3]. The exact solution of (1) found in [4],

$$u(x) = \frac{2}{1 + x^2} \tag{3}$$

corresponds to the one-soliton solution of (2), and its singularities are two poles, $x = \pm \mathrm{i}$. The corresponding problem concerning the singularities of solutions of (1) for $p \neq 2$, to the best of our knowledge, has not been studied. However, the fact that its *localized* solutions, i.e. the solutions obeying the condition

$$\lim_{|x|\to\infty} u(x) = 0 \tag{4}$$

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can be extended from the real axis into the complex domain was proved in [5]. In addition, in [5] it was conjectured that under fairly general conditions the extensions of solutions of the equations of such kind are meromorphic functions or fractional powers of meromorphic functions. We note here that the search for regular ways for the analysis of singularities of solutions of such integro-differential equations can be quite promising bearing in mind a possible extension of the Painlevé approach to such nonlocal problems.

In this paper we study singularities of extensions into the complex domain of solutions of type (4) of equation (1). It is known [6, 7] that a solution of (1), (4) exists as a function in $\mathbb{R}$; in what follows this solution will be denoted by $U_p(x)$ (sometimes we will omit the index p in $U_p(x)$ assuming $p > 1$ to be fixed and integer valued). Below the analytical continuation of $U_p(x)$ into the complex domain is denoted by $U_p(z)$. This paper offers an approach to the analysis of the closest to real axis singularity of $U_p(z)$. It includes analytical and numerical counterparts. In fact, we numerically find the location of the closest to real axis singular point of $U_p(z)$ and show that the behaviour of $U_p(z)$ in the vicinity of this singularity does not correspond to any power of meromorphic function. This approach also can be applied in a more general case when the Hilbert transform in (1) is replaced by a more general Fourier multiplier operator.

Let us now introduce the following notation. We call a singular point $z = z_0$ of a function $\psi(z)$ a *singularity of PP-type* ('a power of pole'), if in some vicinity of $z = z_0$ the function $\psi(z)$ can be represented as a *power* ρ, $\rho \neq -1, -2, \ldots$ *of some meromorphic function which has a simple pole at* $z = z_0$. Evidently, for noninteger ρ the point $z = z_0$ is an algebraic or transcendent branching point of $\psi(z)$. In some neighbourhood of $z = z_0$ the function $\psi(z)$ can be represented by the convergent series

$$\psi(z) = \frac{1}{(z - z_0)^\rho} \sum_{n=0}^{\infty} A_n (z - z_0)^n. \tag{5}$$

The main question we analyse in this paper, which is closely related with the mentioned conjecture can now be formulated as follows: *are all singularities of* $U_p(z)$ *of PP-type?*

Note that, since equation (1) is nonlocal, the direct substitution of (5) into (1) (which allows us to analyse singularities in ODE case; see [8]) cannot be applied.

2. Basic lemma

Let us define the Fourier transform pair in $\mathbb{R}$ by the formulae

$$\hat{u}(\lambda) = \int_{-\infty}^{\infty} u(x) \mathrm{e}^{\mathrm{i}\lambda x} \, \mathrm{d}x$$
$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\lambda) \mathrm{e}^{-\mathrm{i}\lambda x} \, \mathrm{d}\lambda.$$

Then the main tool of our approach is contained in the following lemma.

Basic lemma (I). *Suppose the function* $u(x)$, $x \in \mathbb{R}$, *obeys the conditions:*

(a) $u(x) \in L_1(-\infty, \infty)$;

(b) $u(x)$ *can be continued into the strip* $S_\gamma^+ = \{0 \leqslant \operatorname{Im} z \leqslant \gamma\}$ *in the complex plane in such a way that* S_γ^+ *belongs to one sheet of* $u(z)$, *and for any* y, $0 \leqslant y \leqslant \gamma$,

$$\lim_{x \to \pm\infty} u(x + \mathrm{i}y) = 0 \tag{6}$$

(c) $u(z)$ has only <u>one</u> singular point $z = z_0 = x_0 + \mathrm{i}y_0$, $y_0 < \gamma$ in S_γ^+, and in some neighbourhood of the point $z = z_0$ the function $u(z)$ can be represented as follows:

$$u(z) = \frac{1}{(z - z_0)^\rho}\left[\sum_{n=0}^{N} A_n (z - z_0)^n + r(z - z_0)\right] \tag{7}$$

for some $\rho \in \mathbb{R}$; $A_0, \ldots, A_N \in \mathbb{C}$ and $r(z - z_0) = \mathrm{o}((z - z_0)^N)$ as $z \to z_0$.

Then the asymptotics of the Fourier transform of $u(z)$ for $\lambda \to \infty$ is

$$\hat{u}(\lambda) = 2\pi \mathrm{e}^{\mathrm{i}z_0\lambda} \sum_{n=0}^{N} \frac{A_n \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(n-\rho)}}{\Gamma(\rho - n)} \lambda^{(\rho-n-1)} + \mathrm{o}(\lambda^{(\rho-N-1)} \mathrm{e}^{-y_0\lambda}) \tag{8}$$

for noninteger ρ and

$$\hat{u}(\lambda) = 2\pi \mathrm{e}^{\mathrm{i}z_0\lambda} \sum_{n=0}^{\rho-1} \frac{A_n \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(n-\rho)}}{\Gamma(\rho - n)} \lambda^{(\rho-n-1)} + \mathrm{o}(\mathrm{e}^{-y_0\lambda}) \tag{9}$$

for integer $\rho \geqslant 1$. Here $\Gamma(\xi)$ is Euler's gamma-function.

The statement given above concerns a well known relation between the behaviour of a function in the vicinity of its closest to real axis singular point and the asymptotics of its Fourier transform for large arguments (see, for example, [9], ch 5). However, in the literature we have failed to find this result *exactly* in the form we need and, therefore, we give the proof in the appendix. The present form of the lemma allows one to operate with *asymptotic expansions* for $u(z)$ as $z \to z_0$; the residual term $r(z - z_0)$ can include 'weak' singularities such as logarithmic ones. Evidently, if ρ is noninteger and $u(z)$ is represented in the vicinity of $z = z_0$ by series (5), then this basic lemma provides an *infinite* number of terms in the asymptotics of $\hat{u}(\lambda)$, $\lambda \to \infty$.

The basic lemma (I) admits various generalizations. We give below one more version of this statement concerned with the case that the continuation of $u(x)$ into the complex domain has more than one singular point with the same imaginary part.

Basic lemma (II). *Suppose that for a function $u(x)$, $x \in \mathbb{R}$ the conditions (a) and (b) of basic lemma (I) hold. Suppose also that in S_γ^+ the function $u(z)$ has singular points $z = z_m$, $z_m = x_m + \mathrm{i}\Omega$, $\Omega < \gamma$, $m = 1, 2, \ldots, M$ and in some neighbourhood of the point $z = z_m$ the function $u(z)$ can be represented as follows:*

$$u(z) = \frac{1}{(z - z_m)^{\rho_m}}\left[\sum_{n=0}^{N_m} A_n^{(m)} (z - z_m)^n + r_m(z - z_m)\right] \tag{10}$$

for some $\rho_1, \ldots, \rho_M \in \mathbb{R}$, $N_1, \ldots, N_M$ non-negative integers, $A_n^{(m)} \in \mathbb{C}$, $n = 1, \ldots, N_m$ and $r(z - z_m) = \mathrm{o}((z - z_m)^{N_m})$, as $z \to z_m$, $m = 1, \ldots, M$.

Then the asymptotics of the Fourier transform of $u(z)$ for $\lambda \to \infty$ is the sum of contributions (8), (9) of the singular points $z = z_m$, $m = 1, \ldots, M$.

The proof of this statement repeats the proof of basic lemma (I) with minor modifications.

Summarizing, under certain conditions *infinitely many* terms in asymptotics of Fourier transform $\hat{u}(\lambda)$, $\lambda \to \infty$ are determined by a *few* singularities of $u(z)$. If one denotes the set of all singularities of $u(z)$ by $Z^{(u)}$, then the singularities which provide the *main contribution* to the asymptotics of $\hat{u}(\lambda)$ belong to the set

$$Z_0^{(u)} = \{z \in Z^{(u)}, \operatorname{Im} z = \min_{w \in Z^{(u)}, \operatorname{Im} w > 0} \operatorname{Im} w\}.$$

3. Statements on the localized solution $U(x)$

In what follows we will apply the basic lemmas to the localized solution $U(x)$ of the problem (1), (4). Let us now list known results about the solution $U(x)$ and reveal the gaps between these results and the conditions of the basic lemmas.

The following statements about $U(x)$ are valid:

(A) $U(x)$ exists, is non-negative, and even [7];

(B) $U(x) \sim G/x^2$ as $x \to \pm\infty$ and

$$G = \lim_{x\to\infty} x^2 U(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} U^p(x)\,\mathrm{d}x \tag{11}$$

(see [10]).

(C) The solution $U(x)$ can be analytically continued to some strip $S_\sigma = \{|\mathrm{Im}\, z| < \sigma\}$ in the complex domain, and in this strip for any $|y| < \sigma$

$$\int_{-\infty}^{\infty} |U(x + \mathrm{i}y)|^2\,\mathrm{d}x < \infty \tag{12}$$

(see [10]).

Evidently, (B) implies that $U(x) \in L_1(-\infty,\infty)$. According to (C), the solution $U(x)$ can be continued analytically into some strip $S_\sigma^+ = \{0 \leqslant \mathrm{Im}\, z < \sigma\}$ and in this strip condition (6) holds. However, this strip is not 'wide enough' to include the singularities of $U(z)$, even the ones closest to the real axis, which are contained in $Z_0^{(U)}$. Also, *a priori* we have no information about the number of singularities in $Z_0^{(U)}$.

So we *assume* without proof that the following statements are valid:

(D^+) $U(z)$ can be continued into the strip $S_{\sigma+\varepsilon}^+ = \{0 \leqslant \mathrm{Im}\, z < \sigma + \varepsilon\}$, $\varepsilon > 0$ which contain the set $Z_0^{(U)}$ of closest to real axis singularities of $U(z)$, and condition (6) in $S_{\sigma+\varepsilon}^+ \setminus S_\sigma^+$ remains valid;

(E^+) $Z_0^{(U)}$ consists of a *finite* number of singularities $z_k = x_k + \mathrm{i}\Omega$, $k = 1, \ldots, M$.

In what follows we give a numerical justification of assumptions (D^+) and (E^+).

4. The idea of the method and the first term of the series (13)

Suppose now that all singularities of $Z_0^{(U)}$ are of PP-type, and in some neighbourhood of each of them, $z = z_k$, $k = 1, \ldots, M$ the function $U(z)$ can be represented by the series

$$U(z) = \frac{1}{(z - z_k)^{\rho_k}} \sum_{n=0}^{\infty} A_n^{(k)} (z - z_k)^n. \tag{13}$$

Let us rewrite equation (1) in its Fourier form

$$(|\lambda| + 1)\hat{U}(\lambda) = \widehat{U^p}(\lambda) \tag{14}$$

and calculate the asymptotics of the both parts using the basic lemma statement.

Consider the asymptotical expansion of $\hat{U}(\lambda)$, $\lambda \to \infty$. According to the basic lemma, the singularity $z = z_k$ contributes to the asymptotics of $\hat{U}(\lambda)$ the term

$$C_k(\hat{U};\lambda) = 2\pi \mathrm{e}^{-\Omega\lambda} \mathrm{e}^{\mathrm{i}x_k\lambda} \sum_{n=0}^{\Delta(\rho_k)} \frac{A_n^{(k)} \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(n-\rho_k)}}{\Gamma(\rho_k - n)} \lambda^{\rho_k - n - 1}. \tag{15}$$

Here we introduced the symbol $\Delta(\rho)$ defined as

$$\Delta(\rho) = \begin{cases} \infty & \text{if } \rho \text{ is a noninteger} \\ \rho - 1 & \text{if } \rho \text{ is an integer.} \end{cases}$$

Completely, the asymptotics has the form

$$\hat{U}(\lambda) = \sum_{k=1}^{M} C_k(\hat{U};\lambda) + \mathrm{o}(\lambda^{-\infty}\mathrm{e}^{-\Omega\lambda})$$

if all ρ_k are nonintegers and

$$\hat{U}(\lambda) = \sum_{k=1}^{M} C_k(\hat{U};\lambda) + \mathrm{o}(\mathrm{e}^{-\Omega\lambda})$$

if some of ρ_k are integers.

Consider the asymptotics of the Fourier transform $\widehat{U^p}(\lambda)$. If $p\rho$ is not a negative integer, then the function $U^p(z)$ has the same singular points as $U(z)$, and in some neighbourhood of $z = z_k$ the following expression holds:

$$U^p(z) = \frac{1}{(z-z_k)^{p\rho_k}} \sum_{n=0}^{\infty} B_n^{(k)}(z-z_k)^n \tag{16}$$

where

$$B_n^{(k)} = \sum_{\substack{i_1+i_2+\cdots+i_p=m \\ 0\leqslant i_k\leqslant m}} A_{i_1}^{(k)} A_{i_2}^{(k)} \cdots A_{i_p}^{(k)}. \tag{17}$$

The contribution of the singularity $z = z_k$ to the asymptotics of $\widehat{U^p}(\lambda)$ is

$$C_k(\widehat{U^p};\lambda) = 2\pi\mathrm{e}^{-\Omega\lambda}\mathrm{e}^{\mathrm{i}x_k\lambda} \sum_{n=0}^{\Delta(p\rho_k)} \frac{B_n^{(k)}\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(n-p\rho_k)}}{\Gamma(p\rho_k - n)} \lambda^{p\rho_k-n-1}. \tag{18}$$

Finally

$$\widehat{U^p}(\lambda) = \sum_{k=1}^{M} C_k(\widehat{U^p};\lambda) + \mathrm{o}(\lambda^{-\infty}\mathrm{e}^{-\Omega\lambda})$$

if all $p\rho_k$ are nonintegers and

$$\widehat{U^p}(\lambda) = \sum_{k=1}^{M} C_k(\widehat{U^p};\lambda) + \mathrm{o}(\mathrm{e}^{-\Omega\lambda})$$

if some of $p\rho_k$ are positive integers. Substitute the asymptotical expansions for $\hat{U}(\lambda)$ and $\widehat{U^p}(\lambda)$ into (14). Due to the presence of the factors $\mathrm{e}^{\mathrm{i}x_k\lambda}$ in expressions for $C^k(\hat{U};\lambda)$ and $C^k(\widehat{U^p};\lambda)$ the contributions of each singular point can be treated separately. The comparison of the leading orders of both expansions yields

$$\rho = \frac{1}{p-1} \tag{19}$$

$$A_0 = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{p-1}} \tag{20}$$

and the principal term of the asymptotical expansion for $\hat{U}(\lambda)$, $\lambda \to \infty$ is

$$\hat{U}(\lambda) = \frac{2\pi\lambda^{-\frac{p-2}{p-1}}\mathrm{e}^{-\Omega\lambda}}{\Gamma(\frac{1}{p-1})(p-1)^{\frac{1}{p-1}}} \sum_{k=1}^{M} \mathrm{e}^{\mathrm{i}x_k\lambda} + \mathrm{o}(\lambda^{-\frac{p-2}{p-1}}\mathrm{e}^{-\Omega\lambda}). \tag{21}$$

Note that the asymptotical behaviour of $\hat{U}(\lambda)$ for $\lambda \to \infty$ is oscillatory if $Z_0^{(U)}$ contains at least one singularity with nonzero real part.

5. Numerical solution of equation (1)

Now, let us compare the asymptotic formula (21) with the results of numerical calculation. In order to find the solution $U(x)$ numerically we introduced in (1) a spectral parameter, μ, via the formula

$$U(x) = \mu^{\frac{1}{p-1}} V(x)$$

so that

$$\int_{-\infty}^{\infty} V^2(x)\,\mathrm{d}x = 1. \tag{22}$$

The function $V(x)$ satisfies the nonlinear eigenvalue problem

$$\mathcal{M}V \equiv -\left(\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}x} - 1\right) V = \mu V^p. \tag{23}$$

Solving the problem (22) and (23), we used the *inverse power method* (IPM) [11]. It generalizes the well known method of computing eigenvalues of a symmetric matrix and consists in the following iteration procedure. Given μ_n and $V_n(x)$ at the nth iteration, μ_{n+1} and $V_{n+1}(x)$ at the $(n+1)$th iteration are calculated by:

(i) solving the equation

$$\mathcal{M}\tilde{V}_{n+1} = V_n^p \tag{24}$$

followed by:

(ii) normalizing the solution

$$V_{n+1} = \tilde{V}_{n+1}/\|\tilde{V}_{n+1}\| \qquad \mu_{n+1} = 1/\|\tilde{V}_{n+1}\|$$
$$\|\tilde{V}\| \equiv \left[\int_{-\infty}^{\infty} \tilde{V}^2(x)\,\mathrm{d}x\right]^{1/2}.$$

Since the solitary wave solution is even, we reduce the initial problem to some finite interval $(0, L)$, where L is sufficiently large, assuming that the derivative equals zero at $x = 0$. Instead of a solitary wave solution, we seek *periodic* solutions which either have zero derivative at $x = L$ (our *first* code; p is arbitrary; L is the half-period) or vanish at $x = L$ (our *second* code; P is odd; L is the quarter-period). We introduced a N-node grid on $(0, L)$ for the function $V(x)$; the number of the nodes N was taken to be $2^{14} = 16\,384$ or $2^{15} = 32\,768$. At each IPM iteration, the linear problem (24) was solved using the cosine FFT routine (the first code) or the odd cosine FFT routine (the second code). The accuracy was controlled by comparing the results for various values of L and N obtained by the first and second codes (for p odd). For an additional control of the accuracy of the numerical solution, we used formula (11) and the relation

$$\int_{-\infty}^{\infty} U(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} U^p(x)\,\mathrm{d}x$$

which follows directly from equation (1).

We have found that the above-described algorithm converges quite rapidly and is insensitive to the initial guess profile $V_0(x)$. An illustrative example is given in table 1, where the values of $U_3(0)$ are presented for $N = 16\,384$ and various values of the length L. The first row of the table contains the results obtained by the first code ($\mathrm{d}U_3(L)/\mathrm{d}x = 0$); the second row corresponds to the second code ($U_3(L) = 0$). Switching to $N = 32\,768$ does not affect all the decimals in the table. Thus, one can admit that both the sequences approach (the

Table 1. The values of $U_3(0)$, calculated for $N = 16\,384$ and various values of L by means of the first ($\mathrm{d}U_3(L)/\mathrm{d}x = 0$) and the second codes ($U_3(L) = 0$).

	$L = 100$	$L = 200$	$L = 400$	$L = 800$
The first code	1.890 498	1.890 604	1.890 631	1.890 639
The second code	1.890 712	1.890 659	1.890 646	1.890 643

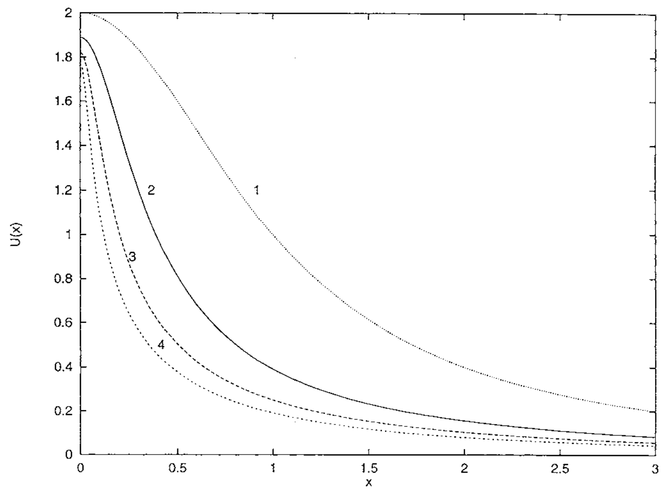

Figure 1. The graphs of the solutions $U_p(x)$ of the problem (1), (4). (Curve 1) $p = 2$ (exact solution $U_2(x) = 2/(1 + x^2)$); (curve 2) $p = 3$; (curve 3) $p = 4$; (curve 4) $p = 5$. Since $U_p(x)$ are even the graphs are depicted for $x \geqslant 0$ only.

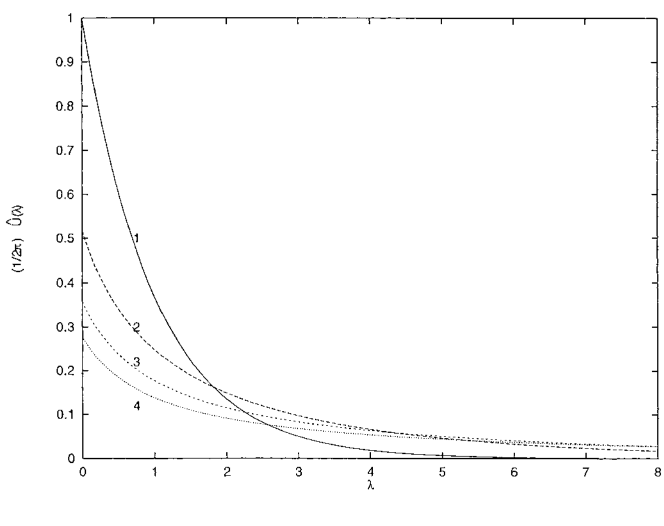

Figure 2. The graphs of the Fourier transforms $\widehat{U_p}(\lambda)$. (Curve 1) $p = 2$, ($\widehat{U_2}(\lambda) = 2\pi \mathrm{e}^{-|\lambda|}$); (curve 2) $p = 3$; (curve 3) $p = 4$; (curve 4) $p = 5$. Since $\widehat{U_p}(\lambda)$ are even the graphs are depicted for $\lambda > 0$.

first one from above, the second one from below) the value $U_3(0) \approx 1.890\,64$, which is exact up to 10^{-5}.

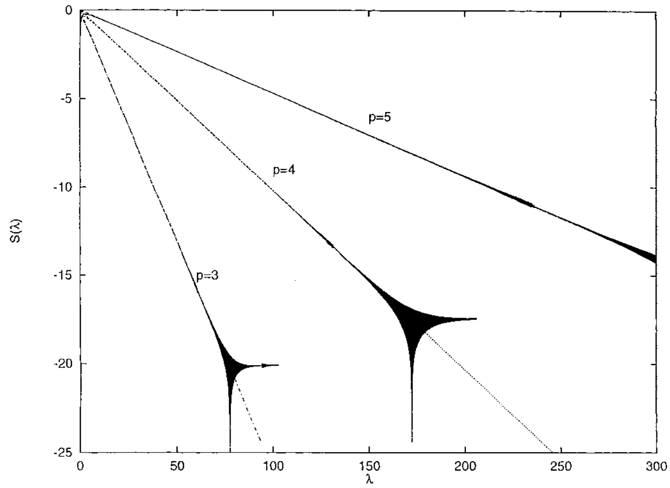

Figure 3. The graphs $S(\lambda)$ and straight lines $S = \Omega_p\lambda$ with correspondingly adjusted coefficient Ω_p, $p = 3, 4, 5$. The black 'tongues' on the graphs correspond to the regions where accuracy of numerical solution is insufficient.

The profiles of the localized solution for $p = 2, 3, 4, 5$ are depicted in figure 1 (since the solutions $U(x)$ are even, the graphs are plotted for $x > 0$ only). Figure 2 represents the Fourier transforms of these solutions. As $\lambda \to \infty$, all the graphs in figure 2 decay without oscillations, so one can expect that for $p = 3, 4, 5$ the set $Z_0^{(U)}$ consists of only a singular point that lies on the imaginary axis, $z = \mathrm{i}\Omega$. In this case formula (21) becomes

$$\hat{U}(\lambda) = \frac{2\pi\lambda^{-\frac{p-2}{p-1}}\mathrm{e}^{-\Omega\lambda}}{\Gamma(\frac{1}{p-1})(p-1)^{\frac{1}{p-1}}} + \mathrm{o}(\lambda^{-\frac{p-2}{p-1}}\mathrm{e}^{-\Omega\lambda}). \tag{25}$$

It corresponds to a one-term asymptotical expansion of $U(z)$ in the vicinity of $z = \mathrm{i}\Omega$:

$$U(z) = \frac{1}{(z-\mathrm{i}\Omega_p)^{\frac{1}{p-1}}}\left[\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{p-1}}}{(p-1)^{\frac{1}{p-1}}} + \mathrm{o}(1)\right]. \tag{26}$$

It follows from (25) that for $\lambda \to \infty$

$$S(\lambda) \equiv \ln\left[\frac{\lambda^{-\frac{p-2}{p-1}}(p-1)^{\frac{1}{p-1}}\Gamma(\frac{1}{p-1})|\hat{U}(\lambda)|}{2\pi}\right] \approx -\Omega\lambda.$$

The graphs of $S(\lambda)$ for $p = 3, 4, 5$ are depicted in figure 3. It is clear from figure 3 that the asymptotical relation $S(\lambda) \approx -\Omega\lambda$ holds perfectly. This makes it possible to calculate numerically the values $\Omega = \Omega_p$ (see table 2). The black 'tongues' in figure 3 indicate the domains where the accuracy of calculation falls off; for example, for $p = 3$ this occurs at $S(\lambda) < -17$ or $|\widehat{U_3}(\lambda)| \sim 10^{-6}$; the order of the latter corresponds to the accuracy of $U_3(x)$ (see above).

A good accordance between the asymptotical formula (25) and the numerical results allows us to suppose that the application of the basic lemma is justified in this case, and in the vicinity of the singularity $z = \mathrm{i}\Omega$ the function $U(z)$ obeys the asymptotic formula (26).

Table 2. The values of $U_p(0)$, $G_\infty \equiv \lim_{x\to\infty} x^2 U_p(x)$ and Ω_p for $p = 2, 3, 4, 5$.

p	$U_p(0)$	G_∞	Ω_p
2	2.000	2.000	1.00
3	1.891	1.034	0.26
4	1.820	0.715	0.10
5	1.772	0.555	0.05

6. The sum of the series (13)

Since both ρ and $p\rho$ are noninteger, all higher-order coefficients A_m of the series (5) can be found in a similar way as the coefficient A_0. Equating the coefficients which correspond to $\lambda^{\rho-m}\mathrm{e}^{-\Omega\lambda}$, $m = 1, 2, \ldots$, of both parts of (14), after some algebra one obtains the following recurrence relation:

$$A_m = \frac{\mathrm{i}}{m+1}\left(A_{m-1} - \sum_{\substack{i_1+i_2+\cdots+i_p=m\\ 0\leqslant i_k<m}} A_{i_1}A_{i_2}\cdots A_{i_p}\right). \tag{27}$$

Series (5) with the coefficients A_m, $m = 0, 1, \ldots$ can be summed explicitly. In order to do this we note that the same formulae (19), (20), (27) will emerge while substituting series (5) into the equation

$$\mathrm{i}\phi_z - \phi + \phi^p = 0 \tag{28}$$

and equating terms with the same power $(z - z_0)^{k-\rho}$, $k = 0, 1, \ldots$. The general solution of (28) is

$$\phi_{\tilde{z}}(z) = (1 - \mathrm{e}^{-\mathrm{i}(p-1)(z-\tilde{z})})^{-\frac{1}{p-1}} \tag{29}$$

where $\tilde{z} \in \mathbb{C}$ is arbitrary. Solution (29) has the singularities $z = \tilde{z} + \frac{2\pi k}{p-1}$, $k \in \mathbb{Z}$ and all of them are of PP-type. The fact that the coefficients of the expansion of (29) at the point $z = \tilde{z}$ coincide with the coefficients A_m, $m = 0, 1, \ldots$ can be checked independently in a straightforward way. So, in some neighbourhood of the singularity $z = z_0$ of $U(z)$ the following equality holds:

$$U(z) = (1 - \mathrm{e}^{-\mathrm{i}(p-1)(z-z_0)})^{-\frac{1}{p-1}}. \tag{30}$$

Formula (30) can be continued analytically to the real axis, and for $z \in \mathbb{R}$ it should provide a solution of (1). However, (30) *does not* satisfy equation (1). So, one can conclude that the singularity $z = z_0$ *is not of PP-type*.

The analytical and numerical results given above can be summarized as follows. The localized solution $U(x)$ of equation (1), being continued in the upper complex half-plane, has a singularity on the imaginary axis, $z = \mathrm{i}\Omega$. In the neighbourhood of this singularity the function $U(z)$ can be represented by the asymptotic expansion

$$U(z) = \frac{1}{(z - \mathrm{i}\Omega)^{\frac{1}{p-1}}}\left[\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{p-1}}}{(p-1)^{\frac{1}{p-1}}} + r(z - \mathrm{i}\Omega)\right]$$

where $r(z - \mathrm{i}\Omega) = \mathrm{o}(1)$, $z \to \mathrm{i}\Omega$. However, the singularity $z = \mathrm{i}\Omega$ is not a singular point of PP-type. So, the function $r(z - \mathrm{i}\Omega)$ cannot be expanded in the Taylor series in $z = \mathrm{i}\Omega$. The problem of more exact description of the behaviour of $U(z)$ in the vicinity of the singularity $z = \mathrm{i}\Omega$ needs further investigation.

7. Discussion: possible generalizations

The approach which we used for the analysis of the singular points of $U(z)$ admits various extensions. With minor modifications it can be applied to the equation

$$\mathcal{L}u = u^p \tag{31}$$

where $\mathcal{L}$ is a Fourier multiplier operator

$$\widehat{\mathcal{L}u}(\lambda) = a(\lambda)\hat{u}(\lambda)$$

and as $\lambda \to \infty$ the symbol $a(\lambda)$ has the asymptotics of the form

$$a(\lambda) = \lambda^\alpha \sum_{m=0}^{\infty} \Lambda_m \lambda^{-m} + \mathrm{o}(\lambda^{-\infty}) \qquad \alpha \in \mathbb{R} \qquad \Lambda_m \in \mathbb{C} \qquad m = 0, 1, \ldots. \tag{32}$$

The procedure of calculation of the coefficients A_m, $m = 0, 1, \ldots$ can be algorithmized and series (5) can be summed up numerically. Comparison of the result of this summation with a straightforward numerical solution of (31) may give an 'empirical' information which may be used for a more rigorous analysis.

Possible modifications of this approach can be generated by various versions of the basic lemma statement. The generalization of the basic lemma to the case of expansions

$$u(z) = \sum_{n=0}^{N} A_n (z - z_0)^{\rho_n} + \mathrm{o}((z - z_0)^{\rho_N}) \rho_0 < \cdots < \rho_N$$

allows one to take into consideration a more general class of nonlinearities, including, for example, polynomials. Another version of the basic lemma links the behaviour of a periodic function $u(z)$

$$u(x + T) = u(x) \tag{33}$$

near its closest to real axis singularities and the asymptotics of the higher coefficients of its Fourier series. This makes it possible to analyse the corresponding problem for periodic solutions in a similar way to the localized ones.

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Appendix. The proof of the basic lemma

In what follows we need the following auxiliary statement [12].

Lemma A. Let $f(x)$, $a \leqslant x \leqslant b$, be a complex-valued function; $0 \leqslant a \leqslant b$, b can be infinite. Let for some $\lambda = \lambda_0$ the integral

$$I(\lambda) = \int_a^b f(x) \mathrm{e}^{-\lambda x} \, \mathrm{d}x \tag{A.1}$$

absolutely converge. Then $|I(\lambda)| \leqslant \mathrm{const} \times \mathrm{e}^{-\lambda a}$, $\lambda > \lambda_0$.

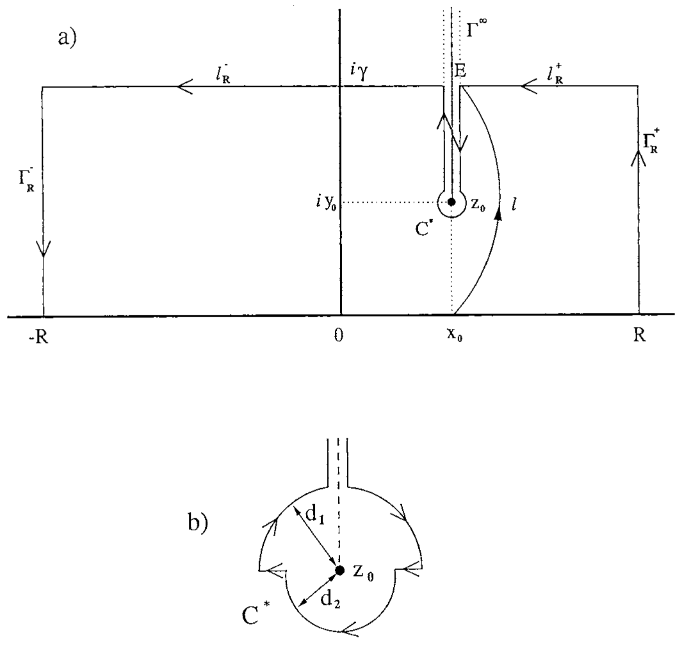

Figure A.1. (*a*) the contour G_R; (*b*) the deformed circuit C^*.

Proof of the basic lemma. Consider the contour G_R depicted in figure A.1(*a*), $G_R = (-R, R) \cup \Gamma_R^+ \cup l_R^+ \cup \Gamma^* \cup l_R^- \cup \Gamma_R^-$; the path Γ^* passes along two sides of the cut E and makes a turn around the point $z = z_0$. The integral of $u(z)\mathrm{e}^{\mathrm{i}\lambda z}$ along G_R vanishes, so

$$\int_{-\infty}^{\infty} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -\int_{\Gamma^*} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z - \lim_{R\to\infty}\left(\int_{\Gamma_R^+} + \int_{l_R^+} + \int_{l_R^+} + \int_{\Gamma_R^-} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z\right). \qquad \text{(A.2)}$$

Below we consider the integrals along the paths $\Gamma_R^\pm$, $l_R^\pm$ and Γ^* consecutively. First of all we note that (6) implies that

$$\lim_{R\to\infty}\int_{\Gamma_R^\pm} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = 0. \qquad \text{(A.3)}$$

The following auxiliary lemma gives an estimate of the integrals along $l_R^\pm$. □

Lemma 1. *If the conditions of the basic lemma hold, then there exists an integer number $k > 0$ such that*

$$\lim_{R\to\infty}\int_{l_R^\pm} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = \mathrm{o}(\lambda^k \mathrm{e}^{-\gamma\lambda}). \qquad \text{(A.4)}$$

Proof of lemma 1. We prove (A.4) for the integral along l_R^+; the arguments for the integral along l_R^- are the same. Consider the closed contour $F = (x_0, R) \cup \Gamma_R^+ \cup l_R^+ \cup l$ (see figure A.1(*a*)) where l is some path which connects points $z = x_0$ and $x_0 + \mathrm{i}\gamma$ in the complex plane, lying entirely in a strip $0 \leqslant \operatorname{Im} z \leqslant \gamma$ and is directed counterclockwise with respect to the singular point $z = z_0$. Since $u(z)$ is analytic inside F, we have

$$\int_{l_R^+} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -\left(\int_{x_0}^{R} + \int_{\Gamma_R^+} + \int_{l} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\right)\mathrm{d}z. \qquad \text{(A.5)}$$

The integrals in the left-hand side of this expression converge uniformly and one can pass to the limit $R \to \infty$ in both parts. Using (A.3), we have

$$\lim_{R\to\infty} \int_{l_R^+} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -\int_{x_0}^{\infty} u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x - \int_{l} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z. \tag{A.6}$$

From the other side

$$\lim_{R\to\infty} \int_{l_R^+} u(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = \mathrm{e}^{-\lambda\gamma}\int_{x_0}^{\infty} u(x+\mathrm{i}\gamma)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x \equiv W(\lambda)\mathrm{e}^{-\lambda\gamma}. \tag{A.7}$$

It follows from (A.6) that the function

$$W(\lambda) \equiv \int_{x_0}^{\infty} u(x+\mathrm{i}\gamma)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x \tag{A.8}$$

is defined and continuous for $\lambda \in \mathbb{R}$ (however, the integral in (A.8) may converge conditionally). Evidently, $W(\lambda)$ is the Fourier transform of $w(x)$ where

$$w(x) = \begin{cases} u(x+\mathrm{i}\gamma) & x > x_0 \\ 0 & x \leqslant x_0 \end{cases} \tag{A.9}$$

and, since $\lim_{x\to\infty} u(x+\mathrm{i}\gamma) = 0$ and $u(x+\mathrm{i}\gamma) \in C^\infty(x_0,\infty)$, $w(x)$ is bounded for $x \in \mathbb{R}$. This implies that $w(x) \in S'(\mathbb{R})$ in the sense of distributions; here $S'(\mathbb{R})$ is the space of distributions of slow growth (see [14]). This means that its Fourier transform $W(\lambda)$ also belongs to $S'(\mathbb{R})$ and there exists an integer constant $k > 0$ such that $W(\lambda) = \mathrm{o}(\lambda^k)$, $\lambda \to \infty$. Together with (A.7) this gives formula (A.4). Lemma 1 is proved. □

Let us turn to the integral along the path Γ^*. Its asymptotics for $\lambda \to \infty$ is governed by the following lemmas 2 and 3.

Lemma 2. *Let Γ^* be the contour shown in figure A.1(a). Then as $\lambda \to \infty$*

$$\int_{\Gamma^*} (z-z_0)^\kappa \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -2\pi\mathrm{e}^{\mathrm{i}\lambda z_0}\frac{\mathrm{e}^{-\pi\mathrm{i}\kappa/2}}{\lambda^{\kappa+1}\Gamma(-\kappa)} + \mathrm{O}(\mathrm{e}^{-\gamma\lambda}) \tag{A.10}$$

if κ is a noninteger, and

$$\int_{\Gamma^*} (z-z_0)^\kappa \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = \begin{cases} -2\pi\mathrm{e}^{\mathrm{i}\lambda z_0}\dfrac{\mathrm{e}^{-\pi\mathrm{i}\kappa/2}}{\lambda^{\kappa+1}\Gamma(-\kappa)} & \kappa < 0 \\ 0 & \kappa \geqslant 0 \end{cases} \tag{A.11}$$

if κ is an integer.

Proof of lemma 2. If κ is integer, then the result (A.11) follows immediately from Cauchy's theorem. Let κ be a noninteger. Complete the contour Γ^* with two half-lines (dashed in figure A.1(*a*)) and denote this new contour by Γ^∞. Evidently,

$$\int_{\Gamma^*} (z-z_0)^\kappa \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = \int_{\Gamma^\infty} (z-z_0)^\kappa \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z + (\mathrm{e}^{2\mathrm{i}\pi\kappa}-1)\int_{x_0+\mathrm{i}\gamma}^{x_0+\mathrm{i}\infty} (z-z_0)^\kappa \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \tag{A.12}$$

where for w^κ we take the main branch, $w^\kappa = |w^\kappa|\mathrm{e}^{\mathrm{i}\kappa\,\mathrm{Arg}w}$, $-\frac{3\pi}{2} < \mathrm{Arg}w \leqslant \frac{\pi}{2}$. After the transformation $z = z_0 - \mathrm{i}z'/\lambda$ the integral along the contour Γ^∞ can be calculated using the integral representation for the gamma function [13]; this gives the first term in the left-hand side of (A.10). After substitution $z = x_0 + \mathrm{i}z'$ the second integral in the left-hand side of (A.12) can be reduced to the form (A.1) and the estimate for the residual term follows from lemma A. This completes the proof of lemma 2. □

Lemma 3. *Let Γ^* be the contour depicted in figure A.1(a) and $q(z-z_0)$ be a function analytic in some punctured neighbourhood of $z = z_0$ and $q(z-z_0) = \mathrm{o}((z-z_0)^\kappa)$ for some κ. Then for $\lambda \to \infty$*

$$\int_{\Gamma^*} q(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = \mathrm{o}\left(\frac{\mathrm{e}^{-y_0\lambda}}{\lambda^{\kappa+1}}\right). \tag{A.13}$$

Proof of lemma 3. The integral in (A.13) can be rewritten as follows:

$$\int_{\Gamma^*} q(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -\int_{x_0+\mathrm{i}(y_0+\theta)}^{x_0+\mathrm{i}\gamma} q_+(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z + \int_{x_0+\mathrm{i}(y_0+\theta)}^{x_0+\mathrm{i}\gamma} q_-(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z + \int_{C^*} q(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \tag{A.14}$$

where $q_\pm(z-z_0)$ are two branches of $q(z-z_0)$ on the sides of the cut E, $\theta > 0$ is some small enough number and C^* is a circuit around the point $z = z_0$ passed in the clockwise direction which starts and finishes at the point $z = z_0 + \mathrm{i}\theta$. Suppose that λ is large enough and choose $\theta = \lambda^{\delta-1}$ where δ is some number, $0 < \delta < 1$. Applying lemma A, we have

$$\left|\int_{x_0+\mathrm{i}(y_0+\lambda^{\delta-1})}^{x_0+\mathrm{i}\gamma} q_\pm(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z\right| = \left|\mathrm{e}^{-\lambda y_0}\int_{\lambda^{\delta-1}}^{\gamma-y_0} q_\pm(\mathrm{i}w)\mathrm{e}^{-\lambda w}\,\mathrm{d}w\right| = \mathrm{O}(\mathrm{e}^{-(y_0\lambda+\lambda^\delta)}). \tag{A.15}$$

To estimate the integral along C^*, we consider the two cases, $\kappa > -1$ and $\kappa < -1$, separately.

$\kappa > -1$. In this case the integrals $\int_{z_0}^{z_0+\mathrm{i}\lambda^{\delta-1}} q_\pm(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z$ converge, and we have

$$\int_{C^*} q(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z = -\int_{z_0}^{z_0+\mathrm{i}\lambda^{\delta-1}} q_+(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z + \int_{z_0}^{z_0+\mathrm{i}\lambda^{\delta-1}} q_-(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z.$$

Denote

$$M_\pm(\lambda) = \sup_{z\in(z_0,z_0+\mathrm{i}\lambda^{\delta-1})}\left|\frac{q_\pm(z-z_0)}{(z-z_0)^\kappa}\right|. \tag{A.16}$$

Since $\lim_{\lambda\to\infty} M_\pm(\lambda) = 0$,

$$\begin{aligned}\left|\int_{z_0}^{z_0+\mathrm{i}\lambda^{\delta-1}} q_\pm(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z\right| &= \left|\int_{z_0}^{z_0+\mathrm{i}\lambda^{\delta-1}}\left[\frac{q_\pm(z-z_0)}{(z-z_0)^\kappa}\right](z-z_0)^\kappa\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z\right| \\ &\leqslant M_\pm(\lambda)\mathrm{e}^{-y_0\lambda}\int_0^{\lambda^{\delta-1}} w^\kappa\mathrm{e}^{-\lambda w}\,\mathrm{d}w \leqslant M_\pm(\lambda)\mathrm{e}^{-y_0\lambda}\int_0^\infty w^\kappa\mathrm{e}^{-\lambda w}\,\mathrm{d}w \\ &= M_\pm(\lambda)\mathrm{e}^{-y_0\lambda}\lambda^{-(\kappa+1)}\Gamma(\kappa+1) = \mathrm{o}\left(\frac{\mathrm{e}^{-y_0\lambda}}{\lambda^{(\kappa+1)}}\right).\end{aligned}$$

Together with (A.15) this formula gives (A.13) for the case $\kappa > -1$.

$\kappa \leqslant -1$. Let λ be large enough. Deform the circuit C^* around z_0 to that formed by two half-circles of radia $d_1 = \lambda^{\delta-1}$ (upper) and $d_2 = \lambda^{-(1+\delta/\kappa)}$ (lower) connected with two transitional paths (see figure A.1(*b*)). So for $z \in C^* \equiv C^*(\lambda)$ we have

$$\operatorname{Im} z \geqslant y_0 - \lambda^{-(1+\delta/\kappa)} \tag{A.17}$$

and consequently

$$|\mathrm{e}^{\mathrm{i}\lambda z}| \leqslant \mathrm{e}^{\lambda^{-\delta/\kappa}-y_0\lambda}. \tag{A.18}$$

Since $q(z-z_0) = \mathrm{o}((z-z_0)^\kappa)$ and $\kappa \leqslant -1$, for $z \in C^*(\lambda)$ there exists $M_1(\lambda) > 0$ such that

$$|q(z-z_0)| < M_1(\lambda)(\lambda^{-(1+\delta/\kappa)})^\kappa. \tag{A.19}$$

As $\lambda \to \infty$ the value $M_1(\lambda) \to 0$ and the contour $C^*(\lambda)$ shrinks to the point $z = z_0$. Evidently, the length of $C^*(\lambda)$ is majorized by $M_2\lambda^{\delta-1}$ with some constant $M_2 > 0$ independent of λ. So one can conclude that

$$\left| \int_{C^*(\lambda)} q(z-z_0)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \right| \leqslant M_2 M_1(\lambda)\mathrm{e}^{\lambda^{-\delta/\kappa}} \left(\frac{\mathrm{e}^{-y_0\lambda}}{\lambda^\kappa} \right) = \mathrm{o}\left(\frac{\mathrm{e}^{-y_0\lambda}}{\lambda^\kappa} \right) \tag{A.20}$$

for $\lambda \to \infty$. This completes the proof of lemma 3. □

Substituting (A.3), (A.4), (A.10), (A.11) and (A.13) into (A.2), we arrive at (8) and (9). The basic lemma is proved.

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